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A Note on State Minimization of a Special Class of Incomplete Sequential Machines

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Abstract—In this note a necessary and sufficient condition is supplied for a flow table to have the property that every cover composed of maximal compatibles is closed. An example of such a flow table is given to which all sufficient conditions known before do not apply.

Index Terms—Incompletely specified machines, maximal compatibles, minimal covers, reduction of incomplete sequential machines, sequential machines, state minimization.

INTRODUCTION

It is well known that state minimization of an incomplete sequential machine becomes considerably easier if there exists a minimal closed cover which is composed entirely of maximal compatibles.[1] This is always true if every collection of maximal compatibles covering the state set is closed. Let E denote the class of all flow tables having the latter property. The theorem given in the following section characterizes the class E by a necessary and sufficient condition that does not seem to be too difficult to check for a given flow table. This condition is in fact more general than the sufficient conditions given by McCluskey [7], Unger [11], and Pager [8]. An example is supplied in what follows for a flow table belonging to the class E to which neither of the above mentioned sufficient conditions does apply.

For a flow table belonging to class E, state minimization reduces to the problem of finding a cover of the state set consisting of a minimum number of maximal compatibles. Such minimal covering problems have been treated by many authors, e.g., Quine [10], McCluskey [6], Grasselli and Luccio [4], Gomory [3], etc. A more advanced contribution to this subject is that of House *et al.* [5]. The covering problem occurring in the connection considered here even bears a special structure: 1) the given sets are all the maximal compatibles of some "compatibility" (i.e., reflexive and symmetric) relation; and 2) the minimality criterion is the number of sets. The author has described a reduction procedure [1], [2] which is applicable exactly to covering problems of this special type.

A CHARACTERIZATION OF CLASS E

The basic facts about incomplete sequential machine minimization are assumed to be known [9]. Let $M=\langle S, X, Y, \delta, \lambda\rangle$ be an incomplete machine with S, X, Y denoting the sets of states, inputs, and outputs, respectively, δ being a partial function from $S\times X$ into S, and λ being a partial function from $S\times X$ into Y. δ is called the next-state function and λ is called the output function. Furthermore, let $\gamma=\{B_1, \cdots, B_m\}$ be the set of all the maximal compatibles. A set of states $I\subset S$ such that there is a $B\in\gamma$ and a $x\in X$ with $I=\delta(B, x)=\{\delta(s, x)|s\in B\}$ is called an *image* of B. Finally let

$$U(s) = \bigcap_{s\in B\in\gamma} B$$

be the intersection of all maximal compatibles containing the state s. Then the following theorem holds.

Theorem: Every cover $\alpha\subset\gamma$ is closed if and only if for each $B\in\gamma$ all images I of B satisfy at least one of the following conditions.

Condition 1: $I\subset B$.

Condition 2: There is a state $s\in S$ such that $I\subset U(s)$.

Proof: To show sufficiency let α be a cover of S, $\alpha\subset\gamma$, and let $B\in\alpha$ and I be an image of B. It has to be shown that

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[1] A *cover* is a collection of subsets of the state set S whose union is S. The definitions of terms *closed* and *maximal compatible* may be taken rom [9, definitions 7 and 11, respectively].

TABLE I

	0	1	2	3	4	5
1	7, 0	2, 0	—, —	8, 1	3, 0	2, 0
2	8, 1	1, —	3, 1	1, 0	—, —	—, —
3	4, 0	7, 1	7, 1	—, —	—, —	5, 0
4	8, 0	8, 1	—, —	—, —	8, 1	—, —
5	—, —	—, —	3, —	6, 0	4, 0	—, —
6	8, 0	9, —	3, 0	—, —	—, —	—, —
7	4, 0	7, 1	—, —	—, —	—, —	—, —
8	4, —	—, —	—, —	—, —	4, 1	—, —
9	7, —	6, —	—, —	—, —	—, —	9, 1

TABLE II

	0	1	2	3	4	5
16	78	29	—	—	—	—
259	78	16	—	16	—	—
289	478	16	—	—	—	—
3478	48	78	—	—	48	—
46789	478	6789	—	—	48	—
357	—	—	37	—	—	—
5679	478	679	—	—	—	—

$I \subset B'$ for some $B' \in \alpha$, i.e., α is closed. If $I \subset B$, then nothing has to be shown. Let $I \not\subset B$. Then by assumption it must be that $I \subset U(s)$ for some $s \in S$. But then $I \subset B'$ for all $B' \in \gamma$ containing s. Since α is a cover, at least one of these sets B' is in α. Necessity is shown indirectly by assuming that the condition of the theorem is not fulfilled. That is, for a $B \in \gamma$ and an image I of B we have $I \not\subset B$ as well as $I \not\subset U(s)$ for all states $s \in S$. Then, for each state $s \in S$, there must exist a maximal compatible $B(s) \in \gamma$ such that $s \in B(s)$, but $I \not\subset B(s)$. Consider the cover

$$\alpha = \{B\} \cup \{B(s) \mid s \in S\}.$$

This cover is not closed, since the image I of B is not contained in any element of α.

Obviously Condition 2 of the theorem holds for all images containing at most one element. Furthermore, if Condition 1 does not hold but Condition 2 does, then the state s such that $I \subset U(s)$ cannot be in B. From that we have the following corollary.

Corollary: Every cover $\alpha \subset \gamma$ is closed if and only if for each $B \in \gamma$ all images I of B such that $|I| \geq 2$ satisfy at least one of the following conditions.

Condition 1: $I \subset B$.

Condition 2': There is a state $s \notin B$ such that $I \subset U(s)$.

An Example

Consider the flow table shown in Table I. The set of maximal compatibles is

$$\gamma = \{16, 259, 289, 3478, 46789, 357, 5679\}.$$

The intersections $U(s)$ for $s = 1, 2, \cdots, 9$ are, respectively

$$16, 29, 37, 478, 5, 6, 7, 8, 9.$$

Table II shows the images of the maximal compatibles under the inputs $0, 1, \cdots, 5$. An entry "—" means that the image has less than two elements. It is easily checked that every image I of any B not contained in B is in one of the intersections $U(s)$, $s \notin B$. The theorem assures that every cover composed of maximal compatibles is closed. Thus, for example, $\mu = \{16, 259, 3478\}$ is a minimal closed cover.

Condition 1 of Pager [8] (equivalently Condition 2) is the most general one of the sufficient conditions we have mentioned in the Introduction, in that it implies all others. It is easily seen that this Condition 1 does not apply to our example: it requires that every image I contains some state which is "dominated" by all the other states of I (state s *dominates* state r, if s is compatible with every state with which r is compatible, i.e., the implication $r \in B \Rightarrow s \in B$ holds for all $B \in \gamma$). This is equivalent to saying that $I \subset U(s)$ for some $s \in I$. However, the image 78 is contained neither in $U(7)$ nor in $U(8)$, but it is contained in $U(4)$.

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