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AN AXIOMATIC APPROACH TO INFORMATION STRUCTURES

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1. Introduction

Information structures are viewed here as formal objects on which certain operations can take place. Three operations are considered basic: selection of parts, construction of new objects by "naming" given ones, and addition of objects, modelling the integration or merging of information structures.

Concrete examples of the sort of objects we have in mind are Vienna objects [6,7] as well as their generalizations and modifications [3,4]. Axiomatic foundations of Vienna objects have been given previously by Ollongren [7] and by the author [3]. Being broader in perspective, however, this paper gives a general axiomatic basis for the type of algebraic structures developed in [4].

2. Basic Operations

Let D be a set, + be a binary operation on D, and 0 be a special object in D. The elements of D will be called (structured) objects. Here is our first axiom:

Addition: $D \times D \to D$

Ax(A) (D,+,0) is a commutative monoid.

We adopt the intuition that objects carry information, and that the addition of objects forms in some sense the union of their information contents. The null object 0 carries empty information.

Selection and construction are accomplished by means of "names" or "access paths", represented by the set of words S^* over a finite alphabet S of selectors. The empty word will be denoted by 1 .

In the sequel, we will maintain the following notational conventions: $a,b,c,\ldots \in D$; $s,t,\ldots \in S$; $x,y,\ldots \in S^*$. Specifications "$\in D$" etc. will be omitted whenever possible. Similarly, if quantification is omitted in subsequent formulas, universal quantification for each free variable occuring should be inserted. In what follows, index i runs over an index set $I \subset \mathbb{N}$.

<u>Selection</u>: $D \times S^* \rightarrow D$

Ax(S1) $a1 = a$

Ax(S2) $a(xy) = (ax)y$

Ax(S3) $(\sum a_i)x = \sum(a_i x)$

<u>Construction</u>: $S^* \times D \rightarrow D$

Ax(C1) $1a = a$

Ax(C2) $(xy)a = x(ya)$

Ax(C3) $x(\sum a_i) = \sum(xa_i)$

Construction xa may be interpreted as giving object a the name x; addition of differently named objects may be viewed as combining them into one composite object, and selection ax selects that part of a named x. The two last axioms require that addition and selection (resp. construction) be independent in the sense that they can be applied in either order.

Algebraically, D is a S^*-left(and right)-semimodule [2]. Right semimodules are called right transformation semigroups in [5].

An object $e \in D$ is called <u>elementary</u> iff, for each $x \neq 1$, we have $ex=0$. Let E be the set of all elementary objects. Its elements will be denoted by letters e,f,... , often omitting the specification "$\in E$". Obviously, E is a subsemigroup of D. Let x and y be called <u>independent</u>, $x \not\sim y$, iff neither is x a prefix of y nor vice versa.

<u>Interaction</u>

Ax(I1) $(xa)x = a$

Ax(I2) $x \not\sim y \Rightarrow (xa)y = 0$

Ax(I3) $\forall a \, \forall x \, \exists b : \; a = x(ax) + b \;\wedge\; bx = 0$

The third axiom claims that each object a can be subdivided into two parts: that part of a named x and the rest b having no x-part.

From these axioms, we can draw several conclusions. Let $A:=D-E$. Obviously, $a \neq 0$ and $x \neq 1$ implies $xa \in A$ since $(xa)x=a\neq 0$. Furthermore, it is easy to prove the following: (i) $xa=xb \Rightarrow a=b$, (ii) $x0=0$, and (iii) $0x=0$. As a consequence of (ii), E is a submonoid of D.

<u>Theorem 2.1</u> : $\forall a \, \exists e : \; a = \sum_{s \in S} s(as) + e$

<u>Proof</u>: We apply axiom (I3) repeatedly. Let $S=\{s_1,\dots,s_n\}$. If $a=a_1$, we have for $i=1,2,\dots,n$:

$$a_i = s_i(a_i s_i) + a_{i+1} = s_i(a\,s_i) + a_{i+1} \quad ,$$

where $a_{i+1}s_k = 0$, $a_{i+1}s_j = a\,s_j$ if $1 \leq k \leq i < j \leq n$. This means that a_{n+1} is elementary . ///

<u>Definition 2.2</u> : $F(a) := \{ e \in E \mid a = \sum_{s \in S} s(as) + e \}$

The sets F(a) play an important role in the sequel. Evidently, $F(a) \neq \emptyset$ for each a, and for elementary objects we have $F(e) = \{e\}$.

Theorem 2.3 : (i) $x \neq 1 \Rightarrow 0 \in F(xa)$

(ii) $F(a) + F(b) \subset F(a+b)$

Proof: (i) is evident from axioms (I1) and (I3). In order to prove (ii), let $e_a \in F(a)$ and $e_b \in F(b)$. Then we have

$$a + b = \sum_{s\in S} s(as) + e_a + \sum_{s\in S} s(as) + e_b$$

$$= \sum_{s\in S} s((a+b)s) + e_a + e_b \qquad ///$$

3. Equality

It is important to have a criterion for the equality of objects. A sufficient condition for finite objects to be equal will be axiomatically extended to arbitrary objects.

Theorem 3.1 : $(\forall s : as = bs \wedge F(a) \cap F(b) \neq \emptyset) \implies a = b$

Proof: $e \in F(a) \cap F(b) \Rightarrow a = \sum_{s\in S} s(as) + e = \sum_{s\in S} s(bs) + e = b$ ///

Definition 3.2 : $a \sim b \;:\Leftrightarrow\; \forall x : F(ax) \cap F(bx) \neq \emptyset$.
If $a \sim b$, a and b will be called compatible .

It is evident that $\sim$ is reflexive and symmetric, and it is an easy exercise to prove that $a \sim b$ implies (i) $a+c \sim b+c$, (ii) $ax \sim bx$, and (iii) $xa \sim xb$. Thus, the transitive closure of $\sim$ is a congruence relation on D.

An object a is called finite iff the set $\{x \mid ax \neq 0\}$ is finite.

Theorem 3.3 : If a and b are finite objects, we have : $a \sim b \Rightarrow a = b$

Proof: Repeated application of theorem 3.1 . ///

These circumstances suggest the last axiom in our axiom system :

Equality
Ax(E) $a \sim b \implies a = b$

This axiom is related to the completeness of data spaces as defined in [1].
As a consequence, each object can be represented by a sum of the following form:

Theorem 3.4 : For each x, let $e_x \in F(ax)$. Then we have

$$a = \sum_{x\in S^*} x\, e_x$$

Proof: Let $y \in S^*$. Since $(\sum_{x\in S^*} xe_x)y = \sum_{z\in S^*} ze_{yz} = e_y + \sum_{z\neq 1} ze_{yz} = e_y + \sum_{s\in S} sb_s$, where $b_s = \sum_{w\in S^*} we_{ysw}$, we have $e_y \in F((\sum xe_x)y) \cap F(ay)$. ///

This theorem allows us to represent each object by a formal power series with coefficients in $\mathbb{P}(E)$, the power set of E:

$$a = \sum_{x\in S^*} x E^x \qquad , \text{ where } E^x = F(ax) \subset E$$

A word x is called <u>peripheral</u> with respect to an object a iff ax is elementary, i.e. $axy=0$ for each $y\neq 1$. Since $F(e) = \{e\}$, the coefficients of peripheral words are singletons.

4. Orthogonality

The considerations made above show that each model of our axiom system can be represented by a set $D\subset\mathbb{P}(E)^{S^*}$ of formal power series, given E and S. The operations are defined in an obvious way, taking account of theorem 2.3 . Adopting the notion of orthogonality as introduced in [1], we define orthogonal sets of formal power series as follows. Here, a(x) denotes the coefficient of x, $a(x)\subset E$.

<u>Definition 4.1</u> : A set $D\subset\mathbb{P}(E)^{S^*}$ is called <u>orthogonal</u> iff

$$\forall \beta \in D^{S^*}\ \exists c \in D\ \forall x \in S^* \ :\ c(x) = \beta(x)(x)$$

A model of our axiom system which is an orthogonal set will be called an orthogonal model.

<u>Theorem 4.2</u> : If D is an orthogonal model, all coefficients are singletons.

<u>Proof</u>: Given a and x , let $\beta\in D^{S^*}$ be any function such that $\beta(x)=a$ and $\beta(xy)=0$ for each $y\neq 1$. By orthogonality, there is an object c such that $c(x)= a(x)$ and $c(xy)= 0$ if $y\neq 1$. Thus, cx is elementary, and we have $|a(x)| = |c(x)| = 1$. ///

As a consequence, each orthogonal model is represented by a subset of $\mathbb{D}=E^{S^*}$. Some of these have been considered in [4]. Vienna objects, however, do not form an orthogonal model: they correspond to the case where each coefficient of a non-peripheral word is the whole set E. Non-peripheral words will be called <u>inner</u> words in the sequel.Thus, in the case of Vienna objects, inner words are not able to carry distinctive information. In order to include this case, we generalize the notion of orthogonality slightly.

<u>Definition 4.3</u> : A function $\beta\in D^{S^*}$ is called <u>admissible</u> iff

$$\forall x\ (\ (\forall s\ :\ \beta(xs)xs = 0\ ) \Longrightarrow |\beta(x)(x)| = 1\ )$$

Let $\mathbb{B}$ be the set of admissible functions in D^{S^*} .

<u>Definition 4.4</u> : A set $D \subset \mathbb{P}(E)^{S^*}$ is called <u>quasi-orthogonal</u> iff

$$\forall ß \in \mathbb{B}\ \exists c \in D\ \forall x \in S^* \ :\ c(x) = ß(x)(x)$$

<u>Theorem 4.5</u> : If D is a quasi-orthogonal model, the coefficients of inner words form a partition on E .

<u>Proof</u>: Let x,y be inner words wrt a resp. b . Let $a(x) \cap b(x) \neq \emptyset$, $e \in E$, $e \neq 0$, and $s \in S$. Due to quasi-orthogonality, objects $c_1 = se + a(x)$ and $c_2 = se + b(y)$ exist by the functions

$$ß_1(z) = \begin{cases} ax & \text{if } z=1 \\ se & \text{if } z=s \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad ß_2(z) = \begin{cases} by & \text{if } z=1 \\ se & \text{if } z=s \\ 0 & \text{otherwise} \end{cases}$$

Evidently, $ß_1$ and $ß_2$ are admissible. By theorem 3.1 , we have $c_1=c_2$ and thus $a(x)=b(y)$. To complete the proof we observe that, for each e, we have $e \in F(se+e)$ by theorem 2.3 . ///

This theorem shows that the set $C = \{ a(x) \mid a \in D \wedge x \in S^* \}$ of all coefficients in a quasi-orthogonal model is of the form $C = C_o \cup C_1$, where C_o is the set of all singletons, and C_1 is a partition on E. From theorem 2.3 (ii) we conclude that the equivalence relation $\equiv$ corresponding to C_1 is a congruence on the monoid (E,+,0). It is not difficult to see that each congruence on (E,+,0) is implied by some quasi-orthogonal model, and that among all quasi-orthogonal models implying a fixed congruence there is one comprehensive model including all others as subsets. This latter model is given by the set of all formal power series associating members of C_o with peripheral words and members of C_1 with inner words. The subset of all finite objects is an example of another model implying the same equivalence relation.

It is shown in [4] that the set of Vienna objects is isomorphic to a quotient structure of the fundamental orthogonal model, provided that there are "no zero sums", i.e. a+b=0 implies a=b=0. This result is generalized in the next theorem.

<u>Theorem 4.6</u> : Let D be a quasi-orthogonal model with no zero sums. Then there is an orthogonal model D' and a congruence relation $\equiv$ on D' such that

$$D \cong D'/\equiv \quad .$$

<u>Proof</u>: Let D' be the set of power series obtained from D by replacing each $a \in D$ by the set of all a' where a' is obtained from a by replacing all coefficients of inner words by arbitrary singletons. It is straightforward to prove that D' is an orthogonal model. Let $\equiv$ be the congruence relation on E implied by the quasi-orthogonality of D. We extend $\equiv$ to D' as follows: for $a,b \in D$, let

$a \equiv b$ iff $a(x) \equiv b(x)$ for all inner words x wrt a and b, and $a(x)=b(x)$ for all peripheral words x wrt a or b. In order to show that $\equiv$ is a congruence on D', we must show that, for each c resp. x, we have $a+c \equiv b+c$, $ax \equiv bx$, and $xa \equiv xb$. The last two relationships are very easy to prove; so we drop it here. That $a+c \equiv b+c$ is seen as follows: the sets of inner resp. peripheral words of a and b are equal. Since there are no zero sums, the set of inner words of a+c is the union of the respective sets of a and c. The same holds true for b. Let x be an inner word wrt a+c. Then, if x is inner wrt a, we have $a(x) \equiv b(x)$; otherwise, we have $a(x)=b(x)$. In each case, we have by theorem 2.3 (ii) : $(a+c)(x)=a(x)+c(x) \equiv b(x)+c(x)=(b+c)(x)$. Now let x be a peripheral word wrt a+c. Then x is peripheral both wrt to a and c, and we have $(a+c)(x)= a(x)+c(x)=b(x)+c(x)=(b+c)(x)$. This shows that $a+c \equiv b+c$.

Now we consider the quotient structure $D'/\equiv$ which is, of course, a model of our axiom system when making the usual conventions about the operations. The isomorphism of $D'/\equiv$ and D is established by the 1-1-correspondence between congruence classes $[a]$ on D' and objects $a' \in D$ given by

$$\forall x \quad : \quad F(a'x) = \bigcup_{b \equiv a} F(bx) \quad . \qquad ///$$

5. Conclusions

The axiom system given here evolved from a specific philosophy of how information is structured in order to be manipulated by computers. It leads to a general framework comprising a great variety of models, from which the Vienna objects and their derivatives considered so far are only special cases. The axiom system is consistent in the sense that there is a model for it. The problems of independence and completeness have not been considered. They lie outside the scope of this paper.

REFERENCES

1. Cremers,A.-Hibbard,T.N.: The semantic definition of programming languages in terms of their data spaces. Proc.4.Fachtagung der GI über Programmiersprachen.Fachberichte Informatik,Springer-Verlag,Berlin 1976, 1-11 .

2. Deussen,P.: Halbgruppen und Automaten. Heidelberger Taschenbücher, Springer-Verlag, Berlin 1971 .

3. Ehrich,H.-D.: Ein axiomatischer Ansatz für eine Algebra strukturierter Objekte. Bericht 5/74, Abteilung Informatik, Universität Dortmund 1974.

4. Ehrich,H.-D.: Outline of an algebraic theory of structured objects. Proc. 3rd International Colloquium on Automata, Languages and Programming, Edinburgh University Press, Edinburgh 1976 .

5. Goodwin,J.R.: The structure of finite right transformation semigroups. Ph.D.thesis, University of California, Berkeley 1969 .

6. Lucas,P.-Lauer,P.-Stigleitner,H.: Method and notation for the formal definition of programming languages. Tech. Report TR 25.087, IBM Lab. Vienna, 1968, rev. 1970 .

7. Ollongren,A.: Definition of programming languages by interpreting automata. Academic Press, London 1974 .